Math 10A with Professor Stankova
Worksheet, Discussion #41; Wednesday, 12/6/2017
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Problems

1. Write the integral $\int_1^3 \frac{dx}{x}$ as a limit of Riemann sums. Write it using 2 intervals using the different methods (left endpoint, right endpoint, midpoint, trapezoid, Simpson's).

Solution: We have that

$$\int_1^3 \frac{dx}{x} = \lim_{n\to\infty} \frac{1}{1+2/n}\cdot\frac{2}{n} + \frac{1}{1+4/n}\cdot\frac{2}{n} + \cdots + \frac{1}{3}\cdot\frac{1}{2n}.$$

And using the methods in order are

$$\frac{1}{1}\cdot 1 + \frac{1}{2}\cdot 1$$

$$\frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1$$

$$\frac{1}{1.5}\cdot 1 + \frac{1}{2.5}\cdot 1$$

$$\frac{1}{2}\left[\frac{1}{1} + 2\cdot\frac{1}{2} + \frac{1}{3}\right]$$

$$\frac{1}{6}\left[\frac{1}{1} + 4\cdot\frac{1}{2} + \frac{1}{3}\right]$$

2. Find $\int x\sqrt{4x+5}dx$.

Solution: We want to use u substitution to simplify this problem. Let $u = 4x+5$. Then $x = \frac{u-5}{4}$ and $dx = \frac{du}{4}$ and so

$$\int x\sqrt{4x+5}dx = \int \frac{u-5}{4}\sqrt{u}\frac{du}{4} = \int \frac{u\sqrt{u} - 5\sqrt{u}}{16}du = \frac{(4x+5)^{5/2}}{40} - \frac{5(4x+5)^{3/2}}{24} + C.$$

3. Find $\int \ln x dx$.

Solution: Use integration by parts. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$. Then

$$\int \ln x dx = x \ln x - \int \frac{xdx}{x} = x \ln x - x + C.$$

4. Find $\int e^x \cos x dx$.

Solution: Let $u = \cos x$ and $dv = e^x dx$. Then $du = -\sin x$ and $v = e^x$. So

$$\int e^x \cos x dx = -e^x \sin x + \int \sin x e^x dx = -e^x \sin x + e^x \cos x - \int \cos x e^x dx$$

So bringing the second integral to the left side gives

$$\int e^x \cos x dx = \frac{e^x \cos x - e^x \sin x}{2} + C.$$

5. Find $\int x^3 e^{-x^2} dx$.

Solution: First we u sub $u = x^2$ to get $du = 2xdx$ and $x^3 dx = (2xdx)(x^2/2) = u/2du$. Therefore

$$\int x^3 e^{-x^2} dx = \int \frac{ue^{-u}}{2} du.$$

Now we integrate by parts to get that

$$\int \frac{ue^{-u}}{2} du = \frac{-ue^{-u} - e^{-u} + C}{2} = \frac{(-x^2 - 1)e^{-x^2} + C}{2}.$$

6. What is the smallest value of n needed to ensure that our numerical approximation method for $\int_1^3 dx/x$ is within $0.0001 = 10^{-4}$ using the different methods?

Solution: In order to do this, you need to set the error bound less than equal to this bound.

$$E_L = \frac{K_1(b-a)^2}{2n} = \frac{1(3-1)^2}{2n} \leq 10^{-4} \implies n \geq 2 \cdot 10^4.$$

$$E_R = \frac{K_1(b-a)^2}{2n} = \frac{1(3-1)^2}{2n} \leq 10^{-4} \implies n \geq 2 \cdot 10^4.$$

$$E_M = \frac{K_2(b-a)^3}{24n^2} = \frac{2(3-1)^2}{24n^2} \leq 10^{-4} \implies n \geq \frac{100}{\sqrt{3}}.$$
$$E_T = \frac{K_2(b-a)^3}{12n^2} = \frac{2(3-1)^2}{12n^2} \leq 10^{-4} \implies n \geq \frac{100\sqrt{2}}{\sqrt{3}}.$$
$$E_S = \frac{K_4(b-a)^5}{180n^4} = \frac{24(3-1)^5}{180n^4} \leq 10^{-4} \implies n \geq \frac{20\sqrt{2}}{\sqrt[4]{15}}.$$

7. Find $\int \frac{dx}{x}$.

Solution: It is $\ln|x| + C$. Technically, we have C_1, C_2 where C_1 is used for $x < 0$ and C_2 is used for $x > 0$ but often we just ignore that and only think about when $C_1 = C_2$.

8. Let a population satisfy the equation $\frac{dN}{dt} = 0.53N$. Find the doubling time.

Solution: Solving, we get that $N(t) = Ce^{0.53t}$. Then the doubling time is τ such that $N(t+\tau) = 2N(t)$ or when $e^{0.53\tau} = 2$ and $\tau = \frac{\ln 2}{0.53}$.

9. A logistic equation is given by $\frac{dP}{dt} = r(1 - P/K)P$. Describe how the solutions depend on the initial value.

Solution: The steady states are $P = 0$ and $P = K$. $P = K$ is stable and $P = 0$ is unstable.

10. Describe the dynamics if there is a constant harvesting occurring and how it depends on h.

Solution: If h is small, then there are two roots as before. At a critical value of h, there is a single steady state which is semistable. For even larger h, then there are no steady states and everything dies in finite time.

11. Solve $\frac{dy}{dt} = -t/y$.

Solution: This is a separable equation and separating gives $ydy = -tdt$ and integrating gives $y^2/2 = -t^2/2 + C$ or $y^2 = C - t^2$ and $y = \sqrt{C - t^2}$. Note that the solution is not just $y = C - t$!

12. When can we compare an integral with $\int_1^\infty \frac{dx}{x^p}$ to show convergence? Divergence? And how?

Solution: We use it to show convergence when $p > 1$, because that is when it converges. To show it, we need to show that $0 \leq \int_1^\infty f(x)dx \leq \int_1^\infty \frac{dx}{x^p}$.

We can use it to show divergence when $p \geq 1$ because that is when it diverges. To show it, we need to show that $\int_1^\infty \frac{dx}{x^p} \leq \int_1^\infty f(x)dx$. Note that we do not have the positivity requirement as in the convergence case..

True/False

13. True **FALSE** One needs to learn the method of mathematical induction in order to find approximations of areas under functions using Riemann sums for a specific n (say $n = 5$).

Solution: For a specific n, we can compute the value.

14. **TRUE** False Antiderivatives are useful in at least three places: solving simple DE's, finding speeds and distances travelled during free-falls, and avoiding using Riemann sums when finding areas after we learn about the Fundamental Theorem of Calculus.

15. **TRUE** False Despite the fact that $(\ln|x|)' = 1/x$ for any $x \neq 0$, the integral $\int 1/xdx$ for $x \neq 0$ is strictly speaking, not equal to $\ln|x| + C$ because the function $1/x$ is discontinuous, causing us to use two different constants for the negative and positive real numbers.

16. True **FALSE** The function e^{-x^2} has no antiderivative in the form of an elementary function because no one has been able to find it.

Solution: The function can be proven not to have one.

17. True **FALSE** Any continuous function on (a, b) is integrable (i.e., it has an antiderivative), but the converse is not true because some continuous functions do not have derivatives.

Solution: The converse is all integrable functions are continuous, which is wrong.

18. **TRUE** False To show that the rule "Integral of a product is the product of integrals" is flawed it suffices to produce one counterexample where it does not work.

19. **TRUE** False The formula for the area of a trapezoid (the product of the average of the bases and the height) can be shown by adding up the areas of the two triangles into which a diagonal divides the trapezoid.

20. **TRUE** False Riemann sums are somewhat cumbersome tools for finding approximations of areas, yet they are absolutely necessary to link antiderivatives to areas.

21. True **FALSE** To calculate the definite integral $\int_{-5}^{5} \sqrt{25 - x^2} dx$, we must find an antiderivative of $\sqrt{25 - x^2}$ and use the FTC I to evaluate it at the ends of the interval $[-5, 5]$.

Solution: We can use the area under the curve definition.

22. **TRUE** False There are at least three ways to compute $\int_{-\pi}^{\pi} \sin(x) dx$.

Solution: We can use the fact that $\sin x$ is odd, find an antiderivative, or use Riemann sums.

23. True **FALSE** Splitting an integral along its interval as in $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ makes sense only when c is between a and b.

Solution: We do not need c to be between a and b.

24. True **FALSE** When we do not know an antiderivative of a function and we cannot find the limit of the corresponding Riemann sums on $[a, b]$, searching for a geometric interpretation of the desired definite integral is also pointless.

Solution: Finding a geometric interpretation may help us actually calculate the integral.

25. **TRUE** False Bounding $f(x)$ on $[a, b]$ from above and below by some constants M and m produces only an estimate of $\int_a^b f(x)dx$.

Solution: It allows us to say that the integral is between $m(b-a)$ and $M(b-a)$, not the exact value.

26. True **FALSE** Geometric areas can be, after all, negative if they appear underneath the x-axis and above some function $f(x)$.

Solution: Geometric area is never negative, but we interpret it as negative for integrals.

27. **TRUE** False We can turn limits of Riemann sums into definite integrals and vice versa.

28. **TRUE** False The formula for the area of a right trapezoid appears in the geometric interpretation of the definite integral $\int_{-12}^{-7} xdx$.

Solution: The area is a trapezoid.

29. **TRUE** False When proving the "baby" definite ILL± and IL*c, we need to descend all the way "down" to limits of Riemann sums, and apply along the way the corresponding LL± and LL*c.

Solution: In order to use those rules, we need to use the definition of definite integrals as limits of Riemann sums and hence LLs.

30. True **FALSE** FTCI says that if you start with a function F(x), then differentiate it, and then integrate it (assuming all these operations are OK), you get back the original function F(x).

> **Solution:** That is FTCII. FTCI says that we can calculate an integral by using an antiderivative.

31. **TRUE** False FTCII says that if you start with a function f(x), integrate f(u) from a to x, and then differentiate with respect to x (assuming all these operations are OK), you will get back the original function f(x).

32. True **FALSE** The "area-so-far" function $F(x) = \int_a^x f(u)\,du$ is > 0 when $f(x)$ increases, and it is < 0 when $f(x)$ decreases.

> **Solution:** It is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$.

33. **TRUE** False A valid PST says that if two functions are equal, then their derivatives are equal and also their integrals are equal (assuming that each exists), so one may attempt to take derivatives (or integrals) on both sides of the equality.

34. **TRUE** False The Substitution Rule is really ILL$\circ$ because it undoes for antiderivatives what CR does for derivatives.

35. True **FALSE** $(\ln|x|)' = 1/|x|$ for all $x \neq 0$.

> **Solution:** The derivative is $1/x$.

36. **TRUE** False Some ideas for substitution that work well in a number of examples are substituting what is under a radical, the denominator of a fraction, and an expression whose derivative appears in the numerator.

37. True **FALSE** Checking your answers after having done substitution is a waste of time.

38. **TRUE** False The "area-so-far" function $F(x) = \int_a^x f(u)\,du$ is concave up where $f(x)$ is increasing, and it is concave down where $f(x)$ is decreasing.

> **Solution:** It is concave up when $F''(x) = f'(x) > 0$, or when f is increasing and concave down when $F''(x) = f'(x) < 0$ or when f is decreasing.

39. **TRUE** False There are two different ways to calculate definite integrals by SR (substitution rule): forget temporarily about the bounds of integration, find an antiderivative, and use FTCI; or go directly forward with SR while not forgetting to change the bounds of integration.

40. True **FALSE** After having done IP (integration by parts), checking your answers by differentiation is a waste of effort since you have already used a valid method(s) to calculate these integrals.

41. **TRUE** False To justify IP on indefinite integrals, we applied PR (product rule) and FTCI.

42. True **FALSE** When deciding which of two functions in an integral $\int h_1(x)h_2(x)\,dx$ will play the role of $u = f(x)$ and which of $v' = g'(x)$ in IP, we follow our intuition because integration is a complicated process and there are no guidelines to follow when doing IP.

> **Solution:** The rule for determining u is: inverse trig functions, logarithms, polynomials, trig functions, exponential functions.

43. **TRUE** False If in the integral $\int h_1(x)h_2(x)\,dx$ we see that $h_1(x)$ has a simpler (or comparable in difficulty) antiderivative while $h_2(x)$'s derivative is simpler than $h_2(x)$, we go for IP with $u = f(x) = h_2(x)$ and $v' = g'(x) = h_1(x)$.

44. True **FALSE** If in the integral $\int h_1(x)h_2(x)\,dx$ each of the functions $h_1(x)$ and $h_2(x)$ has equally complicated derivatives and integrals as itself, then there is no point in applying IP, since it will turn the integral into an equally hard integral.

> **Solution:** Sometimes we have to use integration by parts twice and add the integral back (such as the case in $\int \sin(x)e^x dx$).

45. True **FALSE** When one of the functions $h_1(x)$ and $h_2(x)$ in the integral $\int h_1(x)h_2(x)dx$ is x^2 and we want to solve the problem via IP, then we must set $u = f(x) = x^2$, because if we do $v' = g'(x) = x^2$ this will make $g(x) = x^3/3$, which is more complicated than x^2 and hence it will complicate our problem.

> **Solution:** If we have inverse trigs or logarithms, they are above polynomials so we should those as u first.

46. True **FALSE** The formulas for the error bounds for the various approximation rules for $\int_a^b f(x)dx$ can be used to find the exact errors for these approximations.

> **Solution:** They are bounds and give you an estimate for the error but not the exact one.

47. **TRUE** False The Trapezoidal Rule sum is the average of the Right Endpoint and Left Endpoint sums for $\int_a^b f(x)dx$.

48. True **FALSE** To "estimate an approximation" means to "find out at most how far it can be from the exact value," and hence this is not useful since we either don't know the exact value, or if we knew it, we wouldn't be even approximating, much more so estimating an approximation of it.

49. True **FALSE** Simpson's Rule uses degree 4 polynomials to better approximate the shape of the graph of $f(x)$, as evidenced by the fourth derivative and the 4th power n^4 in the formula for the error of the Simpson's approximation: $|E_S| \leq \frac{K_4(b-a)^5}{180n^4}$.

> **Solution:** It approximates using quadratics.

50. True **FALSE** The larger the difference between the maximum and the minimum of $f'(x)$ on $[a, b]$, the bigger the estimates of the errors for E_L and E_R will turn out to be.

> **Solution:** We are concerned with the maximum of $|f'(x)|$, not the difference between the max and min of $f'(x)$.

51. **TRUE** False To find out how far we have to go with the number of subintervals n of $[a, b]$ in order to ensure that our approximation is close enough to the true value of $\int_a^b f(x)dx$, we need to set up an inequality using a formula for the error bounds and solve it for n.

52. True **FALSE** For the same number n of subintervals, the Midpoint Rule tends to be more precise than the Trapezoidal Rule, but Simpson's Rule is always more precise than the Trapezoidal Rule.

> **Solution:** The first part is true but Simpson's rule is not always more precise (see approximating $|x|$).

53. True **FALSE** If a function is concave down, we can obtain an overestimate by applying either the Right Endpoint Rule or the Left Endpoint Rule.

Solution: If it is concave up/down, we can use midpoint or trapezoid rule. If it is increasing/decreasing, we use left/right endpoint.

54. **TRUE** False Infinitely many continuous functions do NOT have antiderivatives in elementary functions, but they still do have antiderivatives, as shown by using the "area-so-far" function and applying FTC II to it.

Solution: See e^{x^2}

55. True **FALSE** We can solve the exponential growth model DE $y'(t) = ky(t)$ only by guessing that $y(t)$ is an exponential function.

Solution: We can also solve by moving the y to one side and the dt to the other.

56. **TRUE** False The logistic model DE is a modification of the exponential growth model, taking into account that environmental resources may be limited to allow unrestricted exponential growth forever.

57. True **FALSE** To solve the logistic model DE $P'(t) = kP(1 - \frac{P}{K})$, we need to integrate both sides and apply integration by parts on the RHS.

Solution: We use partial fractions on the RHS.

58. **TRUE** False The logistic model DE can be modified to account for a constant harvesting rate h by subtracting h from the RHS of the DE.

59. True **FALSE** The growth rate of $P(t)$ in the logistic model is the logarithmic derivative of $P(t)$.

Solution: The growth rate is $P'(t)$, but the relative growth rate is $P'(t)/P(t)$, which is given by the logarithmic derivative.

60. **TRUE** False The relative growth rate in the exponential decay model remains constant for all t.

61. **TRUE** False The method of separable DE can be applied only when the RHS of a DE $\frac{dy}{dt} = f(y, t)$ can be somehow written as a product of a function in y alone and a function in t alone.

62. True **FALSE** The half-life of a radioactive element during exponential decay depends on the initial amount of this element.

Solution: The half life is constant and hence does not depend on the initial amount.

63. **TRUE** False We can get a pretty good idea of the solutions to the harvesting modification of the logistic model DE $P'(t) = kP\left(1 - \frac{P}{K}\right) - h$ by factoring the quadratic polynomial in P on the RHS and studying where it is positive, negative, or 0.

64. **TRUE** False In the modified the logistic model $\frac{dP}{dt} = kP(t)\left(1 - \frac{P(t)}{K}\right) - h$, there is a value of the harvesting rate h for which $P(t)$ has a unique equilibrium, above which all solutions are decreasing to this equilibrium and below which the population becomes extinct.

65. True **FALSE** We can show that $\int_5^\infty \frac{1}{x^{1.01}} dx$ converges in at least three ways: by a brute force calculation using the definition of an improper integral, by representing $\int_5^\infty \frac{1}{x^{1.01}} dx$ as part of $\int_1^\infty \frac{1}{x^{1.01}} dx$ and then using a formula from class for the value of the latter integral, or by comparing it with the more familiar to us integral $\int_5^\infty \frac{1}{x^1} dx$.

Solution: We cannot compare it to $\int_5^\infty 1/x dx$ because that diverges.

66. **TRUE** False If we cannot compute the exact value of an improper integral $\int_{-\infty}^b f(x)dx$, we could try to compare it with another integral $\int_{-\infty}^b g(x)dx$, but that, if successful, would only tell us if the original integral converges or diverges.

67. **TRUE** False The value of $\int_0^\infty \sin x\, dx$ depends on where we "stop" the variable t when calculating the limit of the proper integrals $\int_0^t \sin x\, dx$.

68. True **FALSE** To show that $\int_0^\infty e^{-x^2}\, dx$ converges, it is enough to compare it with $\int_0^\infty e^{-x}\, dx$.

Solution: We know that $-x \geq -x^2$ only for $[1, \infty)$ so we need to split up the integral first to $\int_0^1$ and $\int_1^\infty$.

69. True **FALSE** If $g(x) \leq f(x)$ on $[a, \infty)$ and $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges.

Solution: This is only true if the latter integral is greater than 0.

70. **TRUE** False Improper integrals in Statistics are used, for example, to compute the areas under probability distribuition functions.

Solution: We use it to show that the area underneath the curve is 1.

71. **TRUE** False The quadratic formula is useful when factoring the RHS of the DE $\frac{dP}{dt} = kP(t)\left(1 - \frac{P(t)}{K}\right) - h$.

72. **TRUE** False A semistable equilibrium is obtained for the modified logistic model when the harvesting constant h is such that the quadratic equation in P, $kP\left(1 - \frac{P}{K}\right) - h$, has a unique root for P.

73. True **FALSE** The value of a convergent two-sided improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ for a continuous function $f(x)$ may depend on where we split the integral as a sum of two one-sided improper integrals $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$; however, the divergence of such an integral does NOT depend on the particular a we choose.

Solution: Both do not depend on the choice of a.

74. **TRUE** False If we already know that $\int_{-\infty}^{\infty} f(x)\,dx$ converges, then we can compute it by choosing symmetric "bus stops"; i.e., as $\lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$; yet, until we know that the integral converges we cannot do this and we must compute instead $\lim_{t\to-\infty} \int_{t}^{0} f(x)\,dx + \lim_{t\to\infty} \int_{0}^{t} f(x)\,dx$.

75. **TRUE** False The improper integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges because the integrand function is even and the integral on the right half on the number line $\int_{0}^{\infty} e^{-x^2}\,dx$ is already shown to converge.

76. **TRUE** False Integrals can be improper in more than two places, but in this class we will concentrate mostly on improper integrals of functions without infinite discontinuities because PDFs will be generally continuous or piecewise continuous.

77. True **FALSE** $\int_{0}^{3} \frac{dx}{x-1} = \ln 2$.

Solution: We cannot use FTCI because the function has a discontinuity on the interval. We cannot use FTCI across these jumps.